Exercise 13

Solve the differential equation.

$$3\frac{d^2V}{dt^2} + 4\frac{dV}{dt} + 3V = 0$$

Solution

This is a linear homogeneous ODE with constant coefficients, so its solutions are of the form $V = e^{rt}$.

$$V = e^{rt} \quad \rightarrow \quad \frac{dV}{dt} = re^{rt} \quad \rightarrow \quad \frac{d^2V}{dt^2} = r^2e^{rt}$$

Plug these formulas into the ODE.

$$3(r^2e^{rt}) + 4(re^{rt}) + 3(e^{rt}) = 0$$

Divide both sides by e^{rt}.

$$3r^2 + 4r + 3 = 0$$

Solve for r.

$$r = \frac{-4 \pm \sqrt{16 - 4(3)(3)}}{2(3)} = \frac{-4 \pm \sqrt{-20}}{6} = -\frac{2}{3} \pm i\frac{\sqrt{5}}{3}$$

$$r = \left\{-\frac{2}{3} - i\frac{\sqrt{5}}{3}, -\frac{2}{3} + i\frac{\sqrt{5}}{3}\right\}$$

Two solutions to the ODE are $e^{(-2/3-i\sqrt{5}/3)t}$ and $e^{(-2/3+i\sqrt{5}/3)t}$. By the principle of superposition, then,

$$\begin{aligned}
V(t) &= C_1e^{(-2/3-i\sqrt{5}/3)t} + C_2e^{(-2/3+i\sqrt{5}/3)t} \\
&= C_1e^{-2t/3}e^{-it\sqrt{5}/3} + C_2e^{-2t/3}e^{it\sqrt{5}/3} \\
&= e^{-2t/3}(C_1e^{-it\sqrt{5}/3} + C_2e^{it\sqrt{5}/3}) \\
&= e^{-2t/3}\left[C_1\left(\cos\frac{\sqrt{5}t}{3} - i\sin\frac{\sqrt{5}t}{3}\right) + C_2\left(\cos\frac{\sqrt{5}t}{3} + i\sin\frac{\sqrt{5}t}{3}\right)\right] \\
&= e^{-2t/3}\left[(C_1 + C_2)\cos\frac{\sqrt{5}t}{3} + (-iC_1 + iC_2)\sin\frac{\sqrt{5}t}{3}\right] \\
&= e^{-2t/3}\left(C_3\cos\frac{\sqrt{5}t}{3} + C_4\sin\frac{\sqrt{5}t}{3}\right),
\end{aligned}$$

where C_1, C_2, C_3, and C_4 are arbitrary constants.